DETECTING CODIMENSION ONE MANIFOLD FACTORS WITH TOPOGRAPHICAL TECHNIQUES

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ABSTRACT. We prove recognition theorems for codimension one manifold factors of dimension $n \geq 4$. In particular, we formalize topographical methods and introduce three ribbons properties: the crinkled ribbons property, the twisted crinkled ribbons property, and the fuzzy ribbons property. We show that $X \times \mathbb{R}$ is a manifold in the cases when X is a resolvable generalized manifold of finite dimension $n \geq 3$ with either: (1) the crinkled ribbons property; (2) the twisted crinkled ribbons property and the disjoint point disk property; or (3) the fuzzy ribbons property.

1. INTRODUCTION

In this paper we provide general position techniques that fully utilize a general position characterization of codimension one manifold factors of dimension $n \geq 4$. A *codimension one manifold factor* is a space X such that $X \times \mathbb{R}$ is a manifold. The famous Cell-like Approximation Theorem of Edwards [1, 4, 7, 8] characterizes the manifolds of dimension $n \geq 5$ as precisely the finite-dimensional resolvable generalized manifolds with the disjoint disk property. In the same vein, it has been shown that codimension one manifold factors of dimension $n \geq 4$ are precisely the finite-dimensional resolvable generalized manifolds with the disjoint concordances property.

However, up until now, practical methods of identifying spaces as codimension one manifold factors have appealed to a weaker general position property, the disjoint homotopies property. How to fully utilize the disjoint concordances property has been somewhat elusive. The ribbons properties introduced in this paper fulfill this role. We show that the ribbons properties, if satisfied by an ANR X, will imply that X has the disjoint concordances property and hence $X \times \mathbb{R}$ has the disjoint disks property. Therefore, a finite-dimensional resolvable generalized manifold X is a codimension one manifold factor if it possesses one of the following: (1) the crinkled ribbons property; (2) the twisted crinkled ribbons property and the disjoint point disk property; or (3) the fuzzy ribbons property. For motivation see the surveys [12]-[15].

2. MANIFOLD FACTORS AND CHARACTERIZATIONS

As previously stated, a space X is a *codimension one manifold factor* if $X \times \mathbb{R}$ is a manifold. The fact that the codimension one manifold factors of finite dimension

$n \geq 4$ are precisely the resolvable generalized manifolds X such that $X \times \mathbb{R}$ has the disjoint disks property follows as a corollary of Edwards' Cell-like Approximation Theorem (cf. [1]). Recall that a space X is said to be *resolvable* if there is a manifold M and a surjective map $f : M \to X$ which is cell-like (i.e., $f^{-1}(x)$ has the shape of a point for all $x \in X$). Moreover, X is said to have the *disjoint disks property* (DDP) if every pair of maps $f, g : D^2 \to Y$ can be approximated by maps that have disjoint images.

Edwards' Cell-like Approximation Theorem states that the manifolds of dimension $n \geq 5$ are precisely the finite-dimensional resolvable generalized manifolds with the disjoint disk property. It is well known that not all resolvable generalized manifolds of dimension $n \geq 5$ have the DDP (cf. [1]). Thus not all resolvable generalized manifolds are manifolds. (In dimension ≤ 2 every generalized manifold is a topological manifold, whereas for the situation in dimensions 3 and 4 see [13]-[15].)

In general, a space X is said to satisfy the (m, n)-*disjoint disks property* $((m, n)$-DDP) if any two maps $f : D^m \to X$ and $g : D^n \to X$ can be approximated by maps with disjoint images. As indicated previously, the $(2, 2)$-DDP is simply called the *disjoint disks property* (DDP). The $(1, 2)$-DDP is called the *disjoint arc-disk property* (DADP). The $(1, 1)$-DDP is called the *disjoint arcs property* (DAP). The $(0, 2)$-DDP is called the *disjoint point-disk property* (DPDP).

All generalized manifolds of dimension $n \geq 3$ are known to have the DAP. A natural question is if the DADP, the middle dimension analogue of the DAP and the DDP, provides a characterization of codimension one manifold factors. As it turns out, the DADP condition is sufficient, but not necessary, to determine if a finite-dimensional resolvable generalized manifold of dimension $n \geq 4$ is a codimension one manifold factor. Examples of codimension one manifold factors of dimension $n \geq 4$ that fail to have the DADP can be found in [1, 6, 11]. In fact some of these examples even fail to have the DPDP.

A list of general position properties that have proved useful in recognizing codimension one manifold factors includes:

- The disjoint arc-disk property [2]
- The disjoint homotopies property [11]
    - The plentiful 2-manifolds property [10]
    - The method of δ-fractured maps [10]
    - The 0-stitched disks property [11]
- The disjoint concordances property [5]

It should be noted here that the disjoint concordances property is the only property listed that provides a characterization of codimension one manifold factors. Specifically, a resolvable generalized manifold X of finite dimension $n \geq 4$ is a codimension one manifold factor if and only if X satisfies the disjoint concordances property.

Definition 2.1. A *path concordance* in a space X is a map $F : D \times I \to X \times I$ (where $D = I = [0, 1]$) such that $F(D \times e) \subset X \times e, e \in \{0, 1\}$. A metric space (X, ρ) satisfies the *disjoint path concordances property (DCP)* if, for any two path homotopies $f_i : D \times I \to X$ $(i = 1, 2)$ and any $\varepsilon > 0$, there exist path concordances $F'_i : D \times I \to X \times I$ such that

$$F'_1(D \times I) \cap F'_2(D \times I) = \emptyset$$

and $\rho(f_i, \mathrm{proj}_X F'_i) < \varepsilon$.

It is the main goal of this paper to establish practical techniques that utilize this property.

In this paper we will be generalizing two properties: the plentiful 2-manifolds property and the method of δ-fractures maps. These properties were developed specifically to detect the disjoint homotopies property in certain settings. We will demonstrate how the analogous ribbons properties can be used to detect the weaker disjoint concordances property.

3. Topographies

We begin by restating the disjoint concordances property from a more functional perspective. In particular, we will restate the disjoint concordance property in terms of the topographies.

Definition 3.1. A *topography* Υ *on* Z is a partition of Z induced by a map $\tau : Z \to I$. The *t-level of* Υ is given by

$$\Upsilon_t = \tau^{-1}(t).$$

Definition 3.2. A *topographical map pair* is an ordered pair of maps (f, τ) such that $f : Z \to X$ and $\tau : Z \to I$. The map f will be referred to as the *spatial map* and the map τ will be referred to as the *level map*. The topography associated with (f, τ) is Υ, where $\Upsilon_t = \tau^{-1}(t)$.

Note that a homotopy $f : Z \times I \to X$ has a naturally associated topography, where $\tau : Z \times I \to I$ is defined by $\tau(x, t) = t$. In particular, we may view $f : Z \times I \to X$ as being equivalent to (f, τ) and we will refer to (f, τ) as the *natural topographical map pair associated with* f.

Definition 3.3. Suppose that for $i = 1, 2$, Υ^i is a topography on Z_i induced by τ_i and $f_i : Z_i \to X$. Then (f_1, τ_1) and (f_2, τ_2) are *disjoint topographical map pairs* provided that for all $t \in I$,

$$f_1(\Upsilon_t^1) \cap f_2(\Upsilon_t^2) = \emptyset.$$

A space X has the *disjoint topographies property* if any two topographical map pairs (f_i, τ_i) $(i = 1, 2)$, where $f_i : D^2 \to X$, can be approximated by disjoint topographical map pairs.

The proof of the following result is straightforward:

Theorem 3.4. *An ANR X has the disjoint topographies property if and only if $X \times \mathbb{R}$ has the disjoint disks property.*

Proof. Suppose X has the disjoint topographies property. For $i = 1, 2$, let $F_i : D^2 \to X \times I$. Let $\text{proj}_X : D^2 \to X$ and $\text{proj}_I : D^2 \to I$ be the standard projection maps. Define $f_i = \text{proj}_X \circ F_i$ and $\tau_i = \text{proj}_I \circ F_i$. Applying the disjoint topographies property we get disjoint topographical map pairs (f_i', τ_i') that are approximations of (f_i, τ_i). Then $F_i' = f_i' \times \tau_i'$ are the desired approximations of F_i with disjoint images.

Suppose that $X \times \mathbb{R}$ has the disjoint disks property. Let (f_i, τ_i) be topographical map pairs for $i = 1, 2$. Then $F_i = f_i \times \tau_i : D^2 \to X \times I$. By the disjoint disks property F_i can be approximated by F_i' with disjoint images. Let $f_i' = \text{proj}_X \circ F_i'$ and $\tau_i' = \text{proj}_I \circ F_i'$. Then (f_i', τ_i') are the desired disjoint topographical map pairs approximating (f_i, τ_i). $\square$

However, this result is not the main focus of this paper. Our aim is to provide alternative equivalent conditions which are more easily verified.

4. Special Category Approximation Properties

We desire to more carefully investigate the disjoint topographies property so as to give it practical utility. Similar to the disjoint homotopies property analyzed in [9], the question of whether a space has the disjoint concordances property ultimately reduces to the following question: given a constant homotopy of a 1-complex and an arbitrary homotopy on another 1-complex, can the natural topographical map pairs associated with these homotopies be adjusted with "control" so as to form disjoint topographical map pairs? In this section, we will clarify these characterizing conditions. In Section 7 we will demonstrate that the conditions give the desired result. The ribbons properties in Sections 8 and 9 will specify practical circumstances in which these conditions may be obtained.

Definition 4.1. A topographical map pair (f, τ) is in the $\mathcal{Z}$ *category* if $f : Z \times I \to X$ and $\tau : Z \times I \to I$ so that $Z \times \{e\} \subset \tau^{-1}(e)$ for $e = 0, 1$. We denote $(f, \tau) \in \mathcal{Z}$.

The $\mathcal{D}$ *category* is defined by letting $Z = D = [0, 1]$. The $\mathcal{K}$ *category* is defined by letting $Z = K$, for some 1-complex.

Definition 4.2. A topographical map pair (f, τ) is in the $\mathcal{Z}_c$ *category* if

(1) $(f, \tau) \in \mathcal{Z}$;
(2) $f : Z \times I \to X$ is a constant homotopy; and
(3) (f, τ) is the natural topographical map pair associated with f.

For emphasis on the relevant characteristics, we define the conditions that will be the main focus of the next section in two stages. In the following definitions, the notation $\mathcal{Z}_i$ is intended to represent a category such as $\mathcal{D}$ or $\mathcal{K}$.

Definition 4.3. A space X has the $\mathcal{Z}_1 \times \mathcal{Z}_2$ *category disjoint topographies property* ($\mathcal{Z}_1 \times \mathcal{Z}_2$ *DTP*) if any two topographical map pairs $(f_i, \tau_i) \in \mathcal{Z}_i$, for $i = 1, 2$, can be approximated by disjoint topographical map pairs $(f_i', \tau_i') \in \mathcal{Z}_i$.

Definition 4.4. A space X has the $\mathcal{Z}_1 \times \mathcal{Z}_2$ *DTP** if for any pair of maps $(f_i, \tau_i) \in \mathcal{Z}_i$, for $i = 1, 2$, there are maps $(f_i', \tau_i') \in \mathcal{Z}_i$ so that each f_i' is an approximation of f_i.

Note specifically that the $\mathcal{Z}_1 \times \mathcal{Z}_2$ *DTP** condition does not require the maps τ_i' to approximate τ_i.

A careful look at the definitions will reveal that the $\mathcal{D} \times \mathcal{D}$ DTP* is just the disjoint concordance property in the language of topographies. Our goal will be to show that the disjoint concordances property is equivalent to more versatile conditions, namely the $\mathcal{K}_c \times \mathcal{K}$ DTP* and the $\mathcal{D}_c \times \mathcal{D}$ DTP* in the case that the target space of the spatial map has the $(0, 2)$-DDP. It is these conditions to which our ribbons properties appeal.

5. Extension Theorems

In this section we recall a couple of classical extension theorems that are used extensively when performing general position adjustments in ANR's. We also establish specific extension theorems applicable in the setting of spaces with the various

disjoint topographies properties. To see a proof of the following homotopy extension theorem the reader can refer to [9].

Theorem 5.1 (Homotopy Extension Theorem (HET)). *Suppose that $f : Y \to X$ is a continuous map where Y is a metric space and X is an ANR, Z is a compact subset of Y and $\varepsilon > 0$. Then there exists $\delta > 0$ such that each $g_Z : Z \to X$ which is δ-close to $f|_Z$ extends to $g : Y \to X$ so that g is ε-homotopic to f. In particular, for any open set U such that $Z \subset U \subset Y$, there is a homotopy $H : Y \times I \to X$ so that:*

(1) *$H_0 = f$ and $H_1 = g$;*
(2) *$g|_Z = g_Z$;*
(3) *$H_t|_{Y-U} = f|_{Y-U}$, for all $t \in I$; and*
(4) *$diam(H(y \times I)) < \varepsilon$ for all $y \in Y$.*

Corollary 5.2 (Map Extension Theorem (MET)). *Suppose that $f : Y \to X$ is a continuous map where Y is a metric space and X is an ANR, Z is a compact subset of Y and $\varepsilon > 0$. Then there exists $\delta > 0$ such that each $g_Z : Z \to X$ which is δ-close to $f|_Z$ extends to $g : Y \to X$ so that $\rho(f, g) < \varepsilon$.*

In the arguments that follow, when we say that "without loss of generality (such and such) maps into an ANR are already adjusted to exploit (some) general position property", we are generally appealing to an application of MET. For example, given maps $f_i : D^2 \to X$, i=1,2, where X is an ANR, with the DAP when we say that we may assume without loss of generality that the restrictions of these maps to a finite (or countable) collection of arcs in the domain have disjoint images, we are applying MET.

Corollary 5.3 (Special DTP Extension Theorem). *Let X be an ANR. Suppose that for $i = 1, 2$, (f_i, τ_i) are topographical map pairs so that $f_i : Y_i \to X$ and $\tau_i : Y_i \to I$, where Y_i is a compact metric space. Suppose further that $A_i \subset Y_i$ is compact so that:*

(1) *$(f_1|_{A_1}, \tau_1|_{A_1})$ and (f_2, τ_2) are disjoint topographical map pairs;*
(2) *(f_1, τ_1) and $(f_2|_{A_2}, \tau_2|_{A_2})$ are disjoint topographical map pairs; and*
(3) *(f_1, τ_1) and (f_2, τ_2) can be approximated by disjoint topographical map pairs.*

Then (f_i, τ_i) can be approximated by disjoint topographical map pairs (f_i', τ_i') so that $(f_i'|_{A_i}, \tau_i'|_{A_i}) = (f_i|_{A_i}, \tau_i|_{A_i})$.

Proof. Suppose the objects in the hypothesis are given. By continuity and local compactness we can find compact neighborhoods N_i of A_i so that (1) and (2) still hold when A_1 is replaced with N_i. Choose $\varepsilon > 0$ so that (1) and (2) still hold with N_i replaced with A_i and (f_i, τ_i) replaced with any ε-approximation of (f_i, τ_i).

Let $\delta_i > 0$ be values promised by the HET for (f_i, τ_i) and choose $\delta > 0$ so that $\delta < \delta_1, \delta_2$. Find δ-approximations (f_i', τ_i') of (f_i, τ_i) that are disjoint topographical map pairs. Let $Z_i = \overline{Y_i - N_i}$, $U_i = Y_i - A_i$, and $g_i = f_i'|_{Z_i}$. Let $f_i' : Y_i \to X$ be the end of the homotopy $H^i : Y_i \times I \to X$ promised by the HET. Then (f_i', τ_i') are the desired disjoint homotopies such that $(f_i'|_{A_i}, \tau_i'|_{A_i}) = (f_i|_{A_i}, \tau_i|_{A_i})$. $\qquad\square$

In the next result, the *end levels of a concordance $F : Y \times I \to X \times I$ or a topographical map pair (f, τ)* defined on $Y \times I$ will refer to $Y \times \{0\}$ and $Y \times \{1\}$.

Propositon 5.4. *Let X be an ANR. Suppose $(f_i, \tau_i) \in \mathcal{Z}$ such that $f_i : Y_i \times I \to X$. If the restriction to the end levels is a disjoint topographical map pair and*

(f_i, τ_i) *can be approximated by disjoint topographical map pairs, then* (f_i, τ_i) *can be approximated by topographical map pairs fixed on the end levels. An analogous result is true for concordances.*

Proof. Let $E_i = Y_i \times \{0, 1\}$. By hypothesis, $(f_i|_{E_i}, \tau_i|_{E_i})$ are disjoint topographical map pairs. Let $\varepsilon > 0$ so that any ε-approximation of $(f_i|_{E_i}, \tau_i|_{E_i})$ are still disjoint topographical map pairs. Let $\delta_i > 0$ be a value promised by the MET for ε and $f_i|_{E_i}$. Choose $\delta > 0$ so that $\delta < \delta_1, \delta_2$. Let (g_i, μ_i) be δ-approximations of (f_i, τ_i) in $\mathcal{Z}$. Then there are ε homotopies between $(f_i|_{E_i}, \tau_i|_{E_i})$ and $(g_i|_{E_i}, \mu_i|_{E_i})$, call these $H^i : E_i \times I \to X$.

For $0 < \zeta < \frac{1}{2}$, let $\theta_\zeta : [0, 1] \to [\zeta, 1 - \zeta]$ be the standard order preserving linear map. Define $f_i^\zeta : Y_i \times I \to X$ such that:

$$f_i^\zeta(x, t) = \begin{cases} H^i((x, 0), \frac{t}{\zeta}) & \text{if } t \in [0, \zeta) \\ g_i(x, \theta_\zeta^{-1}(t)) & \text{if } t \in [\zeta, 1 - \zeta] \\ H^i((x, 1), \frac{1-t}{\zeta}) & \text{if } t \in (1 - \zeta, 1] \end{cases}$$

and define $\tau_i^\zeta : Y_i \times I \to X$ such that:

$$\tau_i^\zeta(x, t) = \begin{cases} \theta_\zeta \mu_i(x, t) & \text{if } t \in [\zeta, 1 - \zeta] \\ t & \text{if } t \in [0, \zeta) \cup (1 - \zeta, 1] \end{cases}$$

Note that $(f_i^\zeta|_{E_i}, \tau_i^\zeta|_{E_i}) = (f_i|_{E_i}, \tau_i|_{E_i})$. Moreover, for sufficiently small ζ, $(f_i^\zeta, \tau_i^\zeta)$ are disjoint topographical map pairs that are an ε-approximation of (f_i, τ_i).

The argument for concordances is analogous. $\square$

6. Tools for Finding Disjoint Topographies

The following four "R" strategies can be used to manipulate topographical map pairs to be disjoint:

(1) *Reimage* - modify the spatial image set;
(2) *Realign* - modify the position of the levels by a self homeomorphism of the domain $D \times I$;
(3) *Reparametrize* - relabel the levels by a continuous map fixing the $t = 0$ and $t = 1$ levels; and
(4) *moRph* - redefine the topographical structure.

The first strategy is realized by adjusting the spatial maps. The last three are realized by adjusting the level maps. It is the fourth strategy that is unique to the topographies approach, adding flexibility in that the shape of the levels can be changed. This is the maneuver that puts the topographical approach at an advantage over the homotopies approach in detecting codimension one manifold factors. It is this last strategy that will be fully exploited by the new ribbons properties of Sections 8 and 9.

This section will be devoted to adapting several basic tools that are useful in constructing approximating disjoint topographical map pairs. The first two results are generalizations of results obtained for homotopies found in [9].

Definition 6.1. Suppose for $i = 1, 2$ that (f_i, τ_i) are topographical map pairs having topographies Υ^i. Then the *set of parameterization points of intersection*, denoted by $\text{PPIN}((f_1, \tau_1), (f_2, \tau_2))$ is

$$\text{PPIN}((f_1, \tau_1), (f_2, \tau_2)) = \{(t_1, t_2) \in I^2 \mid f_1(\Upsilon_{t_1}^1) \cap f_2(\Upsilon_{t_2}^2) \neq \emptyset\}.$$

In the next result we show that if $PPIN((f_1, \tau_1), (f_2, \tau_2))$ is 0-dimensional, then we may obtain approximating disjoint topographical map pairs by reparametrizing the levels of the topography. In particular, a reparametrization is a relabeling of the levels determined by replacing t with a function $\gamma(t)$.

Lemma 6.2 (Reparametrization Lemma). *Suppose for $i = 1, 2$ that (f_i, τ_i) are topographical map pairs having topographies Υ^i such that $PPIN((f_1, \tau_1), (f_2, \tau_2))$ is 0-dimensional and $f_1(\Upsilon_e^1) \cap f_2(\Upsilon_e^2) = \emptyset$, for $e = 0, 1$. Then there are arbitrarily close approximations τ_i' of τ_i so that (f_1, τ_1') and (f_2, τ_2') are disjoint topographical map pairs.*

Proof. Suppose $\varepsilon > 0$. Since $Z = PPIN((f_1, \tau_1), (f_2, \tau_2))$ is 0-dimensional there is a path $\gamma : I \to I \times I - Z$ from $(0, 0)$ to $(1, 1)$ such that $|\gamma(t) - t| < \varepsilon$. Let $\tau_i' = \gamma \circ \tau_i$. Then (f_i, τ_i') are disjoint topographical map pairs. $\square$

Propositon 6.3. *Suppose X is a locally compact ANR with the DAP. Then X has the disjoint topographies property if and only if for any pair of topographical maps (f_i, τ_i), for $i=1,2$ such that $f_i : D \times I \to X$, there exist arbitrarily close approximations (f_i', τ_i'), such that $PPIN((f_1', \tau_1'), (f_2', \tau_2'))$ is 0-dimensional.*

Proof. To see the forward direction, assume without loss of generality, that

$$f_1(D \times \{0, 1\} \cup \{0, 1\} \times I) \cap f_2(D \times \{0, 1\} \cup \{0, 1\} \times I) = \emptyset$$

where $D \times I = [0, 1] \times [0, 1]$. It follows from the hypothesis that the collection of maps $(f_1', \tau_1', f_2', \tau_2')$ such that (f_1', τ_1') and (f_2', τ_2') are disjoint topographical map pairs is dense in $\mathcal{D} \times \mathcal{D}$ and this collection is clearly open by continuity arguments. Let $\gamma_k : I \to I$ be a countable collection of maps such that the complement of the images in the interior of $I \times I$ is 0-dimensional and $\gamma_k(e) = e$ for $e = 0, 1$. Find approximations (f_i', τ_i') so that (f_1', τ_1') and $(f_2', \gamma_k \tau_2')$ are disjoint topographical map pairs for all k. Then $PPIN((f_1', \tau_1'), (f_2', \tau_2'))$ is 0-dimensional.

The reverse direction follows almost immediately from the Reparametrization Lemma. The only technicality is that we need to satisfy $\Upsilon_e^1 \cap \Upsilon_e^2 = \emptyset$. We may modify (f_i, τ_i) by assuming that τ_i is a piecewise linear general position map with care taken so that $\Upsilon_e^i = D \times \{e\}$ for $e = 0, 1$. Then we apply the DAP and the MET to modify f_i so that $f_1(\Upsilon^1) \cap f_2(\Upsilon^2) = \emptyset$. $\square$

Propositon 6.4. *Let X be a finite-dimensional ANR with the $(m_1 - 1, m_2)$-DDP and the $(m_1, m_2 - 1)$-DDP. Suppose that (f_i, τ_i) are topographical map pairs such that $f_i : Y_i \to X$ and $\tau_i : Y_i \to X$, where Y_i is a k-complex such that $k \leq m_i$. Then there exist approximations (f_i', τ_i') that are disjoint topographical map pairs.*

Proof. Begin by modifying the maps $\tau_i : Y_i \to I$, if necessary, so that each level is a $(k-1)$-complex. This can be accomplished by approximating τ_i by a piecewise linear map in general position. Next, apply the $(m_1 - 1, m_2)$-DDP and the $(m_1, m_2 - 1)$-DDP conditions to adjust the maps f_i so that each rational level of f_1 is disjoint from the image of f_2 and each rational level of f_2 is disjoint from the image of f_1. Denote the adjusted maps by (f_i', τ_i'). Then $PPIN((f_1', \tau_1'), (f_2', \tau_2'))$ is closed 0-dimensional. If follows by the Reparametrization Lemma that (f_i', τ_i') can be approximated by disjoint topographical map pairs. $\square$

7. Equivalence Theorem

In this section, we will demonstrate the following equivalence theorem:

Theorem 7.1 (Equivalence Theorem). *Let X be a locally compact separable ANR with the DAP. Consider the statements:*

(a) *X has the $\mathcal{D}_c \times \mathcal{D}$ DTP*.*
(b) *X has the $\mathcal{K}_c \times \mathcal{K}$ DTP*.*
(c) *X has the $\mathcal{D} \times \mathcal{D}$ DTP*.*
(d) *X has the disjoint concordance property.*
(e) *$X \times \mathbb{R}$ has the disjoint disks property.*

Then (b)-(e) are equivalent. If in addition, X has the $(0,2)$-DDP, then (a)-(e) are equivalent.

Proof. Observe that (c) and (d) are trivially equivalent since the $\mathcal{D} \times \mathcal{D}$ DTP* is the disjoint concordance property in the language of topographies. In particular, equate (f_i, τ_i) as a topographical map pair with $F_i = f_i \times \tau_i$ as a concordance. The fact that (d) and (e) are equivalent was the main result established in [4]. The fact that (c) implies (a) is trivial since $\mathcal{D}_c \subset \mathcal{D}$.

It suffices to show that: (e) implies (b); (b) implies (c); and (a) implies (b) in the case that X has the $(0,2)$-DDP.

(e) $\implies$ **(b)**: In a locally compact separable ANR, the DDP condition is equivalent to having the property that any two maps $\lambda_i : P_i \to X$ can be approximated by maps with disjoint images where P_i are 2-complexes (see [1, Proposition 24.1]). Given topographical map pairs $(f_1, \tau_1) \in \mathcal{K}_c$ and $(f_2, \tau_2) \in \mathcal{K}$ we may assume without loss of generality, by applying the DAP, that the restrictions to the end levels are disjoint topographical map pairs. By Proposition 5.4, there are approximations (f_i', τ_i') that are disjoint topographical map pairs fixed on the end levels. These are the desired approximations.

(b) $\implies$ **(c)**: Let $(f_i, \tau_i) \in \mathcal{D}$ for $i = 1, 2$. According to [9, Theorem 3.3] there are piecewise linear approximations τ_i' of τ_i such that there exist:

(1) a collection of 1-complexes $K_1^i, \ldots, K_n^i$; and
(2) a collection maps $\phi_j^i : K_i^i \times [t_{j-1}^i, t_j^i] \to D^2$ so that:
   (a) $\bigcup \mathrm{im}(\phi_j^i) = D^2$;
   (b) $\phi_j^i : K_j^i \times [t_{j-1}^i, t_j^i] \to D^2$ is an embedding away from $K_j^i \times \{t_{j-1}^i, t_j^i\}$; and
   (c) $\tau_i' \circ \phi_j^i$ is a level preserving map.

Without loss of generality we may assume that $t_j^1 = t_j^2$ by subdividing into smaller intervals if necessary. Thus we will denote $t_j = t_j^1 = t_j^2$.

Denote $L_j^i = \tau_i^{-1}(t_j)$. These 1-complexes are called the transition levels. Note that

$$L_0^i = \phi_1^i(K_1^i \times \{t_0\}) \text{ and } L_n^i = \phi_n^i(K_n^i \times \{t_n\})$$

and for $j = 1, \ldots, n-1$,

$$L_j^i = \phi_j^i(K_j^i \times \{t_j\}) \cup \phi_{j+1}^i(K_{j+1}^i \times \{t_j\}).$$

By applying the DAP, we may assume that $f_1(L_j^1) \cap f_2(L_j^2) = \emptyset$.

Without loss of generality we may assume that the subintervals $[t_{j-1}, t_j]$ are sufficiently small so that the adjustments that will now follow will also be small.

We will begin by modifying the maps ϕ_j^i so that the $\mathcal{K}_c \times \mathcal{K}$ DTP* condition may be exploited. For $j = 1, \ldots n$, let $s_j = \dfrac{t_{j-1} + t_j}{2}$. Define maps:

(1) $$\theta_j^1 : K_j^1 \times [t_{j-1}, s_j] \to D^2; \theta_j^1(z, t) = \phi_j^1(z, 2t - t_{j-1})$$

(2) $$\lambda_j^1 : K_j^1 \times [s_j, t_j] \to D^2; \lambda_j^1(z, t) = \phi_j^1(z, t_j)$$

(3) $$\theta_j^2 : K_j^2 \times [t_{j-1}, s_j] \to D^2; \theta_j^2(z, t) = \phi_j^2(z, t_{j-1})$$

(4) $$\lambda_j^2 : K_j^2 \times [s_j, t_j] \to D^2; \lambda_j^2(z, t) = \phi_j^2(z, 2t - t_j)$$

Consider the natural topographical map pairs $(f_1\theta_j^1, \eta_j^1)$, $(f_1\lambda_j^1, \mu_j^1)$, $(f_2\theta_j^2, \eta_j^2)$, and $(f_2\lambda_j^2, \mu_j^2)$ defined for these homotopies, respectively. By applying the $\mathcal{K} \times \mathcal{K}_c$ DTP* condition and Proposition 5.4 on $(f_1\theta_j^1, \eta_j^1)$ and $(f_2\theta_j^2, \eta_j^2)$ we can find approximations $(g_j^1, \tilde{\eta}_j^1)$ and $(g_j^2, \tilde{\eta}_j^2)$, respectively, that are disjoint topographical map pairs fixed on the end levels. Likewise, there are approximations of $(f_1\lambda_j^1, \mu_j^1)$ and $(f_2\lambda_j^2, \mu_j^2)$, namely $(h_j^1, \tilde{\mu}_j^1)$ and $(h_j^2, \tilde{\mu}_j^2)$, respectively, that are disjoint topographical map pairs also fixed on the end levels.

Let

$$f_i'(x) = \begin{cases} g_j^i(\phi_j^i)^{-1}(x) & \text{if } \tau'(x) \in [t_{i-1}, s_i] \\ h_j^i(\phi_j^i)^{-1}(x) & \text{if } \tau'(x) \in [s_i, t_i] \end{cases}$$

and

$$\tau_i'(x) = \begin{cases} \tilde{\eta}_j^i(\phi_j^i)^{-1}(x) & \text{if } \tau'(x) \in [t_{i-1}, s_i] \\ \tilde{\mu}_j^i(\phi_j^i)^{-1}(x) & \text{if } \tau'(x) \in [s_i, t_i]. \end{cases}$$

Then (f_i', τ_i') are the desired disjoint topographical map pairs in $\mathcal{D}$ that are approximations of (f_i, τ_i).

(a) $\implies$ (b): Note that this is the only case that requires the $(0, 2)$-DDP condition. Let $(f_1, \tau_1) \in \mathcal{K}_c$ and $(f_2, \tau_2) \in \mathcal{K}$, where $f_i : K_i \times I \to X$ and $\tau_i : K_i \times I \to I$. Let A_i be the 1-complex $[K_i \times \{0, 1\}] \cup [K_i^{(0)} \times I]$. By applying the DAP and Corollary 6.4 (using the DAP and the $(0, 2)$-DDP) we may assume without loss of generality that:

(1) $(f_1|A_1, \tau_1|A_1)$ and (f_2, τ_2) are disjoint topographical map pairs;
(2) (f_1, τ_1) and $(f_2|A_2, \tau_2|A_2)$ are disjoint topographical map pairs;
(3) the restriction of f_1 to $K_1 \times \{0\}$ is an embedding;
(4) the restriction f_2 to A_2 is an embedding; and
(5) $f_1(A_1) \cap f_2(A_2) = \emptyset$.

We wish to define topographical maps $(g_i, \eta_i) \in \mathcal{D}$ that can guide the appropriate modifications of (f_i, τ_i) to give approximations that are disjoint topographical map pairs. To this end, using also the DAP, we may find $\alpha_i : D \to X$ so that:

(6) $f_i(K_i \times \{0\}) \subset \alpha_i(D)$;
(7) $\alpha_1(D) \cap \alpha_2(D) = \emptyset$; and
(8) α_i is a piecewise linear embedding, and in particular the restriction of α_i to $\alpha_i^{-1}f_i\left((\sigma - \sigma^{(0)}) \times \{0\}\right)$ is an embedding for each simplex $\sigma \in K_i$.

The maps α_i will determine the 0-level maps of the new maps $g_i : D \times I \to X$ which we will now construct. For reference in D, let $Q_i = \alpha_i^{-1}(f_i(K_i \times \{0\}))$ and $P_i = \alpha_i^{-1}(f_i(K_i^{(0)} \times \{0\}))$. For reference in $D \times I$, define $B_i = Q_i \times \{0, 1\} \cup P_i \times I$ and $C_i = Q_i \times I$.

Define $g_1 : D \times I \to X$ to be the constant homotopy so that $g_1(x, t) = \alpha_1(x)$. Let $\eta_1 : D \times I \to I$ be the standard projection map. Define $\tilde{g}_2 : D \times \{0\} \cup C_2 \to X$ so that:

$$\tilde{g}_2(x, t) = \begin{cases} \alpha_2(x) & \text{if } t = 0 \\ f_2(z, t) & \text{if } x \in Q_2 \text{ and } \alpha_2(x) = f_2(z, 0) \end{cases}$$

Likewise, define $\tilde{\eta}_2 : D \times \{0, 1\} \cup C_2 \to I$ such that:

$$\tilde{\eta}_2(x, t) = \begin{cases} t & \text{if } t = 0, 1 \\ \tau_2(z, t) & \text{if } x \in Q_2 \text{ and } \alpha_2(x) = f_2(z, 0) \end{cases}$$

Since X is an ANR, $\tilde{g}_2$ extends to a map $g_2 : D \times I \to X$ and $\tilde{\eta}_2$ extends to a map $\eta_2 : D \times I \to I$. Note that we have used sufficient care in our construction so that:

(1) $(g_1|_{B_1}, \eta_1|_{B_1})$ and $(g_2|_{C_2}, \eta_2|_{C_2})$ are disjoint topographical map pairs; and
(2) $(g_1|_{C_1}, \eta_1|_{C_1})$ and $(g_2|_{B_2}, \eta_2|_{B_2})$ are disjoint topographical map pairs.

By the $\mathcal{D}_c \times \mathcal{D}$ DTP*, (g_i, η_i) can be approximated by disjoint topographical map pairs. Hence we also have that

(3) $(g_i|_{C_i}, \eta_i|_{C_i})$ can be approximated by disjoint topographical map pairs.

Therefore, by applying the Special DTP Extension Theorem, there are disjoint topographical map pairs (h_i, μ_i) that are approximations of $(g_i|_{C_i}, \eta_i|_{C_i})$ so that $(h_i|_{B_i}, \mu_i|_{B_i}) = (g_i|_{B_i}, \eta_i|_{B_i})$. This determines approximations (f_i', τ_i') of (f_i, τ_i) that are disjoint topographies. In particular, $(f_i'(z, t), \tau_i'(z, t)) = (h_i(x, t), \mu_i(x, t))$, where $f_i(z, 0) = \alpha_i(x)$.

$\square$

Remark 7.2. Note that it is an easy matter to show that conditions (a)-(d) imply the DAP. Given two singular arcs in X, use these paths to define constant path homotopies and apply any one of the conditions to approximate by disjoint topographical map pairs in the case of (a)-(c) or disjoint concordances in the case of (d). The end levels provide the disjoint approximations. The equivalence of (e) with (a)-(d) does require the DAP.

8. Crinkled Ribbons Properties

Recall that, given $k \geq 0$, a subset $Z \subset X$ of space X is said to be *locally k-coconnected* (k-LCC) if for every point $x \in X$ and every neighbourhood $U \subset X$ of x, there exists a neighbourhood $V \subset U$ of x such that the inclusion-induced homomorphism $\pi_k(V \setminus Z) \to \pi_k(U \setminus Z)$ is trivial. Also recall the following useful Proposition (see [1, Corollary 26.2A]):

Propositon 8.1. *Each k-dimensional closed subset A of a generalized n-manifold X, where $k \leq n - 2$, is 0-LCC.*

We are now ready to define the ribbons properties.

Definition 8.2. A generalized n-manifold X has the *crinkled ribbons property (CRP)* provided that any constant homotopy $f : K \times I \to X$, where K is a 1-complex can be approximated by a map $f' : K \times I \to X$ so that:

(1) $f'(K \times \{0\}) \cap f'(K \times \{1\}) = \emptyset$; and
(2) $\dim(f'(K \times I)) \leq n - 2$.

Theorem 8.3. *If X is a resolvable generalized n-manifold, $n \geq 4$, with the crinkled ribbons property, then X has the $\mathcal{K}_c \times \mathcal{K}$ DTP*.*

Proof. Let $(f_1, \tau_1) \in \mathcal{K}_c$ and $(f_2, \tau_2) \in \mathcal{K}$. Apply the hypothesis of the theorem to find $f_1' : K_1 \times I \to X$ so that $f_1'(K_1 \times \{0\}) \cap f_1'(K_1 \times \{1\}) = \emptyset$, and $\dim(f_1'(K_1 \times I)) \leq n - 2$. It follows that $f_1'(K_1 \times I)$ is 0-LCC in X. Let $A_0 = f_1'(K_1 \times \{0\})$ and $A_1 = f_1'(K_1 \times \{1\})$. Define $\tau_1' : K_1 \times I \to I$ so that:

$$\tau_1'(x, t) = \frac{d(f_1'(x, t), A_0)}{d(f_1'(x, t), A_0) + d(f_1'(x, t), A_1)}.$$

Apply the 0-LCC condition to approximate f_2 by $f_2' : K_2 \times I \to X$ so that

$$f_2'(K_2 \times [\mathbb{Q} \cap I] \cup \mathbb{K}_2 \times I) \cap f_1'(K_1 \times I) = \emptyset,$$

where $\mathbb{K}_2$ is a countable dense set in K_2 containing the vertex set. Then $(f_2')^{-1}(f_1'(K_1 \times I))$ is closed 0-dimensional set Z. Approximate τ_2 so that τ_2' is 1-1 on Z. Then $\mathrm{PPIN}((f_1', \tau_1'), (f_2', \tau_2'))$ is a closed 0-dimensional set. By the Reparametrization Lemma, (f_i', τ_i') can be approximated by disjoint topographical map pairs. $\qquad\square$

Corollary 8.4. *If X is a resolvable generalized n-manifold, $n \geq 4$, with the crinkled ribbons property, then $X \times \mathbb{R}$ has the disjoint disks property.*

Proof. Follows directly from Theorem 8.3 and the Equivalence Theorem. $\qquad\square$

Definition 8.5. A generalized n-manifold X has the *twisted crinkled ribbons property (CRP-T)* provided that any constant homotopy $f : D \times I$ can be approximated by a map $f' : D \times I$ so that:

(1) $f'(D \times \{0\}) \cap f'(D \times \{1\})$ is a finite set of points; and
(2) $\dim(f'(D \times I)) \leq n - 2$.

Theorem 8.6. *If X is a generalized n-manifold of dimension $n \geq 4$ having the twisted crinkled ribbons property and the property that points are 1-LCC embedded in X, then $X \times \mathbb{R}$ has the $\mathcal{D}_c \times \mathcal{D}$ DTP*.*

Proof. It suffices to show that maps in $\mathcal{D}_c \times \mathcal{D}$ can be approximated by disjoint topographical map pairs.

Let $(f_1, \tau_1) \in \mathcal{D}_c$ and $(f_2, \tau_2) \in \mathcal{D}$. Since any generalized manifold of dimension ≥ 3 has the DAP, we may assume without loss of generality that (f_i, τ_i) are disjoint on the end levels, (i.e., $f_1(D \times \{e\}) \cap f_2(D \times \{e\}) = \emptyset$ for $e = 0, 1$), and that any adjustments hereafter are sufficiently small to maintain this condition. Apply the hypothesis of the theorem to find $f_1' : D \times I \to X$ so that $f_1'(D \times \{0\}) \cap f_1'(D \times \{1\})$ is a finite set of points P, and $\dim(f_1'(D \times I)) \leq n - 2$. It follows that $f_1'(D \times I)$ is 0-LCC in X. We may also apply the hypothesis that points are 1-LCC embedded in X and assume without loss of generality that $f_2(D \times I) \cap P = \emptyset$.

Choose $\zeta > 0$ so that $d(f_2(D \times I), P) > \zeta$. Let $A_0 = f_1'(D \times \{0\}) - N(P, \zeta)$ and $A_1 = f_1'(D \times \{1\}) - N(P, \zeta)$. Define

$$\tau_1' : D \times \{0, 1\} \cup (D \times I - (f_1')^{-1}(N(P, \zeta))) \to I$$

so that:

$$\tau_1'(x, t) = \begin{cases} e & \text{if } t = e \\ \dfrac{d(f_1'(x, t), A_0)}{d(f_1'(x, t), A_0) + d(f_1'(x, t), A_1)} & \text{otherwise.} \end{cases}$$

Since $D \times I$ is an AR, τ_1' may be extended to all of $D \times I$. Apply the 0-LCC condition to approximate f_2 by $f_2' : D \times I \to X$ so that:

$$f_2'(D \times [\mathbb{Q} \cap I] \cup [\mathbb{Q} \cap D] \times I) \cap f_1'(D \times I) = \emptyset.$$

Then $(f_2')^{-1}(f_1'(D \times I))$ is a closed 0-dimensional set Z. Approximate τ_2 so that τ_2' is 1-1 on Z. Then $\mathrm{PPIN}((f_1', \tau_1'), (f_2', \tau_2'))$ is a closed 0-dimensional set. By the Reparametrization Lemma, (f_i', τ_i') can be approximated by disjoint topographical map pairs. $\qquad \square$

Corollary 8.7. *If X is a generalized n-manifold of dimension $n \geq 4$ having the twisted crinkled ribbons property and the property that points are 1-LCC embedded in X, then $X \times \mathbb{R}$ has the disjoint disks property.*

Proof. The assertion follows directly from Theorem 8.6 and the Equivalence Theorem. Note that the condition that points are 1-LCC embedded implies the $(0, 2)$-DDP. $\qquad \square$

Remark 8.8. Not all generalized manifolds of dimension $n \geq 4$ have the property that points are 1-LCC embedded. For example, the Daverman-Walsh 2-ghastly spaces are resolvable generalized manifolds that do not have the $(0, 2)$-DDP, and hence cannot satisfy the condition that points are 1-LCC embedded [6].

The following corollary was also proved in [1, 2] by using shrinking techniques. This is the first time general position arguments have been applied to this setting.

Corollary 8.9. *If X is a resolvable generalized locally spherical n-manifold, $n \geq 4$, then X is a codimension one manifold factor.*

Proof. The locally spherical condition implies the twisted crinkled ribbons property. To see this, let $f : D \times I$ be a constant homotopy. Cover the image of f by small neighborhoods $B_1, B_2, \ldots B_n$ so that ∂B_i is an embedded $(n-1)$-sphere. Approximate f by a constant path homotopy $f' : D \times I \to \bigcup \partial B_i$. Without loss of generality we may assume that there are $t_i \in D$ such that $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ and $f'([t_{i-1}, t_i] \times I) \subset \partial B_n$. Since ∂B_i is an $(n-1)$-sphere, f' can be approximated by $f'' : D \times I \to \bigcup \partial B_i$ such that the restriction of f'' to $\bigcup(t_{i-1}, t_i) \times I$ is an embedding and $f'' = f'$ on $\{t_0, t_1, \ldots t_n\} \times I$. Then f'' is the desired approximation of f. $\qquad \square$

9. Fuzzy Ribbons property

The fuzzy ribbons property is the most remarkable generalization of the disjoint homotopies techniques. In particular the fuzzy ribbons property is a generalization of the method of δ-fractured maps. Recall that

Definition 9.1. A map $f : D \times I \to X$ is said to be δ-*fractured* over a map $g : D \times I \to X$ if there are pairwise disjoint balls $B_1, B_2, \ldots, B_m$ in $D \times I$ such that:

    (1) $diam(B_i) < \delta$;
    (2) $f^{-1}(im(g)) \subset \bigcup_{i=1}^m int(B_i)$; and
    (3) $diam(g^{-1}(f(B_i))) < \delta$.

However, because of the freedom in defining the level map to obtain the DTP* conditions, we need no longer require δ-control. The analogous definition in the setting of topographical map pairs is therefore:

Definition 9.2. Let $(f_i, \tau_i) \in \mathcal{K}$ be such that $f_i : K_i \times I \to X$ and $\tau_i : K_i \times I \to I$. Then (f_2, τ_2) is said to be *fractured* over a topographical map pair (f_1, τ_1) if there are disjoint balls $B_1, B_2, \ldots, B_m$ in $K_2 \times I$ such that:

(1) $f_2^{-1}(im(f_1)) \subset \bigcup_{j=1}^{m} int(B_i)$; and

(2) $\tau_1 \circ f_1^{-1} \circ f_2(B_i) \neq I$.

We are now ready to define the fuzzy ribbons property:

Definition 9.3. A space X has the *fuzzy ribbons property (FRP)* provided that for any topographical map pairs, $(f_1, \tau_1) \in \mathcal{K}_c$ and $(f_2, \tau_2) \in \mathcal{K}$, and $\varepsilon > 0$ there are maps τ_i' and ε-approximations f_i' of f_i so that (f_2', τ_2') is fractured over (f_1', τ_1').

Theorem 9.4. *If a space X is an ANR with the DAP having the fuzzy ribbons property, then X has the $\mathcal{K}_c \times \mathcal{K}$ DTP*.*

Proof. Let $(f_1, \tau_1) \in \mathcal{K}_c$ and $(f_2, \tau_2) \in \mathcal{K}$ such that $f_i : K_i \to X$. Using the DAP we may assume without loss of generality that $f_1(K_1 \times I) \cap f_2(K_2 \times \{0, 1\} \cup K_2^{(0)} \times I) = \emptyset$. Apply the fuzzy ribbons property to obtain maps τ_i' and approximations f_i' of f_i so that (f_2', τ_2') is fractured over (f_1', τ_1'). The approximations of f_i should be sufficiently small so that $f_1'(K_1 \times I) \cap f_2'(K_2 \times \{0, 1\} \cup K_2^{(0)} \times I) = \emptyset$. Then there are disjoint balls $B_1, B_2, \ldots, B_m$ in $K_2 \times I - K_2 \times \{0, 1\} \cup K_2^{(0)} \times I$ such that:

(1) $(f_2')^{-1}(im(f_1')) \subset \bigcup_{j=1}^{m} int(B_i)$; and

(2) $\tau_1' \circ (f_1')^{-1} \circ f_2'(B_i) \neq I$.

For each $j = 1, \ldots, m$, choose $t_j \in I - \tau_1' \circ (f_1')^{-1} \circ f_2'(B_i)$. Now define $\tau_2'' : K_2 \times \{0, 1\} \cup K_2^{(0)} \times I \cup (\bigcup B_j) \to I$ so that

$$\tau_2''(x, t) = \begin{cases} t & \text{if } t = 0, 1 \text{ or } x \in K_2^{(0)} \\ t_j & \text{if } (x, t) \in B_j \end{cases}$$

Extend τ_2'' to $K_2 \times I$. Then (f_1', τ_1') and (f_2', τ_2'') are the desired disjoint topographical map pairs. $\square$

Corollary 9.5. *If a space X is an ANR with the the FRP, then $X \times \mathbb{R}$ has the DDP.*

Proof. The DAP follows from the FRP. The rest follows directly from Theorem 9.4 and the Equivalence Theorem. $\square$

Remark 9.6. Certain 2-ghastly spaces satisfy the FRP, such as those discussed in [10]. The same type of arguments apply, however less attention to control is needed to satisfy the FRP.

10. Epilogue

The DTP* properties presented in this paper are not only more versatile in detecting codimension one manifold factors, they also provide a characterization of such spaces. The ribbons properties represent practical applications of these properties. Further interesting questions that may be investigated using the DTP* or ribbons properties include:

Question 10.1. *If G is an $(n-2)$-dimensional cell-like decomposition of an n-manifold M, where $n \geq 4$, is M/G a codimension one manifold factor?*

Question 10.2. *Is every Busemann G-space of dimension $n \geq 5$ a manifold? Equivalently, are small metric spheres in these spaces codimension one manifold factors?*

Question 10.3. *Is every finite-dimensional resolvable generalized manifold of dimension $n \geq 4$ a codimension one manifold factor?*

Acknowledgments

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